Power Inequalities and Spectral Dominance of Generalized Matrix Norms

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Dedicated to Alston S. Householder
on the occasion of his seventy-fifth birthday.

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ABSTRACT

A generalized matrix norm G dominates the spectral radius for all $A\in M_n(C)$ (i) if for some positive integer k the rule $G(A^k)\leqslant G(A)^k$ holds for all $A\in M_n(C)$ and (ii) if and only if for each $A\in M_n(C)$ there exists a constant γ_A such that $G(A^k)\leqslant \gamma_A G(A)^k$ for all positive integers k. Other results and examples are also given concerning spectrally dominant generalized matrix norms.

A generalized matrix norm (gmn) is a map $G:M_n(C)\to R$ which, for all $A,B\in M_n(C)$, satisfies

$$G(A)\geqslant 0, \quad \text{and} \quad G(A)=0 \text{ if and only if } A=0; \tag{1}$$

$$G(cA)=|c|G(A) \qquad \text{for all complex numbers } c; \tag{2}$$

and

$$G(A+B)\leqslant G(A)+G(B). \tag{3}$$

Thus a gmn is simply a vector norm on $M_n(C)$ considered only as an n^2-dimensional vector space. Denote the set of eigenvalues of $A\in M_n(C)$ by

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$\sigma(A)$; then the *spectral radius* $\rho(A)$ of A is defined by

$$\rho(A) \equiv \max_{\lambda \in \sigma(A)} |\lambda|.$$

We call a gmn G *spectrally dominant at* $A \in M_n(C)$, a local concept, if

$$\rho(A) \leqslant G(A). \tag{4}$$

Furthermore, if (4) holds for all $A \in M_n(C)$, we simply say that G is *spectrally dominant*. Since it has no prescribed correlation with the multiplicative structure of $M_n(C)$, a gmn may or may not be spectrally dominant. It is our interest to determine circumstances under which a gmn is spectrally dominant.

A gmn G is said to be *multiplicative* if for all $A, B \in M_n(C)$,

$$G(AB) \leqslant G(A)G(B), \tag{5}$$

and a multiplicative gmn is standardly called a *matrix norm*. It is well known that multiplicativity, when imposed upon a gmn, is a condition sufficiently strong to imply spectral dominance [3, 4, 6]. However, a gmn may be spectrally dominant without being multiplicative, and it is our goal to give conditions sufficient for spectral dominance which amount to weakenings of (5). One of these is sufficiently weak to provide a characterization of spectral dominance. Statement (5) implies, for example, that

$$G(A^2) \leqslant G(A)^2, \tag{6}$$

and further that

$$G(A^k) \leqslant G(A)^k \tag{7}$$

for all $A \in M(C)$ and all positive integers k. We call statements such as these *power inequalities* or *integral power rules* and note below that a gmn may satisfy rules such as these without being multiplicative.

EXAMPLE 1. On $M_2(C)$, define G by

$$G\begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \max\{|a|, |d|\} + |b| + \tfrac{1}{2}|c|.$$

Then it may be verified inductively that G satisfies (7) for each positive integer k and all $A \in M_2(C)$. The lack of multiplicativity of G is apparent from consideration of

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and it is straightforward to verify that G satisfies the requirements of a gmn.

A well-known example of a gmn which is not a matrix norm is the *numerical radius* r defined [2] by

$$r(A) = \max_{x^*x=1} |x^*Ax|.$$

The numerical radius is spectrally dominant and satisfies (7) for all positive integers k and all $A \in M_n(C)$.

It is well known that the spectral radius ρ is not itself a gmn, but a stronger statement can be made.

REMARK. *There exists no generalized matrix norm G which is a function only of the eigenvalues and the Jordan canonical form of its argument.*

Proof. Suppose that G is such a gmn. Then it suffices to consider G restricted to $M_2(C)$, and

$$G\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} = G\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

will hold for all $k > 0$. But then

$$\begin{aligned}(k+1)G\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &= G\begin{pmatrix} 0 & (k+1) \\ 0 & 0 \end{pmatrix} = G\left[\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}\right] \\ &\leqslant G\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} + G\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} = G\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix} + G\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \\ &= 2G\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\end{aligned}$$

and

$$G\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \leqslant \frac{2}{k+1} G\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

for all $k>0$. This implies

$$G\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}=0,$$

a contradiction which means there is no such G.

If G is a gmn on $M_n(C)$, then

$$\mathfrak{B}_G \equiv \{A \in M_n(C) : G(A) \leqslant 1\}$$

is a compact set with the zero matrix as an interior point. Since ρ is a continuous function on $M_n(C)$, it attains a maximum $m(G)>0$ on $\mathfrak{B}_G$:

$$m(G) \equiv \max_{G(A) \leqslant 1} \rho(A). \tag{8}$$

Thus $\rho(A) \geqslant m(G)$ implies $G(A) \geqslant 1$, and $\rho(A) \geqslant 1$ implies $G(A) \geqslant 1/m(G)$. We call $m(G)$ the *spectral characteristic* of G. Because of its homogeneity (2), G is spectrally dominant if $m(G) \leqslant 1$, and we call G *minimally spectrally dominant* if $m(G)=1$. Any gmn G which is not spectrally dominant may be made spectrally dominant by a scalar multiplication; in particular, $\tilde{G} \equiv m(G)G$ is minimally spectrally dominant, and in general $m(tG)=m(G)/t$ for a positive constant t.

THEOREM 1. *The spectral characteristic m is a convex function of its argument.*

Proof. Suppose G_1 and G_2 are two arbitrary gmn's on $M_n(C)$, and $0<\alpha<1$. By definition,

$$m(\alpha G_1 + (1-\alpha) G_2) = \max \frac{\rho(A)}{\alpha G_1(A) + (1-\alpha) G_2(A)}$$

and

$$\alpha m(G_1) + (1-\alpha) m(G_2) = \max \frac{\alpha \rho(A)}{G_1(A)} + \max \frac{(1-\alpha)\rho(A)}{G_2(A)},$$

where all maxima are taken over $A \neq 0$. Since

$$\max\left[\frac{\alpha\rho(A)}{G_1(A)} + \frac{(1-\alpha)\rho(A)}{G_2(A)}\right]$$

is less than or equal to this latter expression, it suffices to show that

$$\frac{\rho(A)}{\alpha G_1(A)+(1-\alpha)G_2(A)} \leqslant \frac{\alpha\rho(A)}{G_1(A)} + \frac{(1-\alpha)\rho(A)}{G_2(A)}$$

for all $0 \neq A \in M_n(C)$. However, this inequality is equivalent to

$$\left[1-\alpha^2-(1-\alpha)^2\right]G_1(A)G_2(A) \leqslant \alpha(1-\alpha)\left[G_1(A)^2+G_2(A)^2\right],$$

which in turn is equivalent to

$$0 \leqslant \left[G_1(A)-G_2(A)\right]^2,$$

so that m is a convex function. ■

COROLLARY 1. *The set of spectrally dominant generalized matrix norms is convex.*

Proof. Immediate. ■

We now turn our attention to relating integral power rules to spectral dominance.

LEMMA 1. *If G is a gmn and $A_1, A_2, \ldots$ is a sequence in $M_n(C)$ such that*

$$\rho(A_j)=1, \qquad j=1,2,\ldots,$$

then $G(A_j) \to 0$ as $j \to \infty$ is not possible.

Proof. Suppose $A_1, A_2, \ldots$ is a countable sequence of the sort assumed and that $G(A_j) \to 0$ as $j \to \infty$. Now $\rho(B) \geqslant 1$ implies $G(B) \geqslant 1/m(G) > 0$. However, $\rho(A_j)=1$, while $G(A_j) < 1/m(G)$ for some j, a contradiction. We conclude that $G(A_j) \to 0$ as $j \to \infty$ is not possible. ■

THEOREM 2. *Let G be a gmn on $M_n(C)$. If there is a constant γ_A (depending only on G and A) such that for all integers $k>0$,*

$$G(A^k) \leqslant \gamma_A G(A)^k, \tag{9}$$

then G is spectrally dominant at A.

Proof. Suppose that γ_A exists with the assumed properties for $A \in M_n(C)$, that (without loss of generality) $\rho(A)=1$, but that the negation of the asserted conclusion holds, namely: $G(A)<1$. Since $G(A)<1$, we have $G(A)^k \to 0$, $\gamma_A G(A)^k \to 0$, and therefore $G(A^k)\to 0$, while $\rho(A^k)=1$ for all $k>0$. Application of Lemma 1 yields a contradiction which means that $\rho(A) \leqslant G(A)$ must hold, and the proof is complete. ∎

A different application of Lemma 1 shows that another type of integral power rule implies spectral dominance.

THEOREM 3. *If G is a gmn on $M_n(C)$ such that for some (fixed) positive integer k,*

$$G(A^k) \leqslant G(A)^k \tag{10}$$

for all $A \in M_n(C)$, then G is spectrally dominant.

Proof. Repeated application of (10) yields

$$G(A^{k^l}) = G\left[(A^{k^{l-1}})^k\right] \leqslant G(A^{k^{l-1}})^k \leqslant \cdots \leqslant G(A)^{k^l}.$$

Suppose (without loss of generality) that $\rho(A)=1$ while $G(A)<1$. Then $\rho(A^{k^l})=1$ for all l, while $G(A)^{k^l}\to 0$ and thus $G(A^{k^l})\to 0$ as $l\to\infty$. Since A is arbitrary, application of Lemma 1 yields a contradiction, which means that G is spectrally dominant. ∎

EXAMPLE 2. For each positive integer k, there exists an n and a spectrally dominant gmn G on $M_n(C)$ such that the rule (10) does not hold for all $A \in M_n(C)$. In fact the set of all gmn's G for which (10) holds for some fixed k is not convex (as is the set of spectrally dominant gmn's). Let G_1 be the maximum absolute row sum norm on $M_n(C)$, and let G_2 be defined by $G_2(A)=G_1(DAD^{-1})$, where D is the diagonal matrix $\operatorname{diag}\{1,2,4,\dots,2^{n-1}\}$. Both G_1 and G_2 are matrix norms and therefore spectrally dominant. Define $G=\frac{1}{2}(G_1+G_2)$, so that G is also spectrally dominant, and let E be the basic n-by-n nilpotent matrix ($e_{ij}=0$ unless $j=i+1$, in which case $e_{ij}=1$). Then $G(E)=\frac{3}{4}$ while $G(E^k)=(2^k+1)/2^{k+1}$ for $k\leqslant n-1$. Thus $N(E)^k=3^k/2^{2k} \ngeqslant N(E^k)$ for $2\leqslant k\leqslant n-1$. In Example 3 below another type of example with such features is given.

For completeness we relate certain cases of equality.

THEOREM 4. *Suppose that G is a gmn on $M_n(C)$. (i) If $A \in M_n(C)$ is such that $G(A^k)=G(A)^k$ for all positive integers k, then $G(A)=\rho(A)$. (ii) On* *the other hand, for any fixed k for which G obeys* (10) *for all* $A \in M_n(C)$, *it follows that* $\rho(A) = G(A)$ *implies* $G(A^k) = G(A)^k$.

Proof. To prove (i), first suppose that $G(A) > \rho(A)$ and, without loss of generality, that $G(A) = 1$. Then $\rho(A) < 1$ so that $A^k \to 0$ as $k \to \infty$. This contradicts $G(A^k) = 1$ for all k. Alternatively, if $G(A) < \rho(A)$, assume without loss of generality that $\rho(A) = 1$. Then $\rho(A^k) = 1$ for all k while $G(A^k) \to 0$ as $k \to \infty$, a contradiction by Lemma 1.

The hypothesis of statement (ii) implies by Theorem 3 that G is spectrally dominant, and the proof of that statement then follows from the observation that

$$\rho(A)^k = \rho(A^k) \leqslant G(A^k) \leqslant G(A)^k = \rho(A)^k. \qquad \blacksquare$$

REMARK. In particular it follows from Theorem 4 that for a gmn G satisfying (10) for all positive integers k and all $A \in M_n(C)$, $\rho(A) = G(A)$ if and only if $G(A^k) = G(A)^k$ for all positive integers k. For example, any matrix norm is such a G, as are gmn's such as the numerical radius. Thus, for example, our Theorem 4 implies Theorem 2 of [2]. Note also that Example 4 below shows that the hypothesis of part (ii) of Theorem 4 may not be weakened to the assumption that G is spectrally dominant.

We now turn to extending Theorem 2 to a characterization of spectral dominance. A sequence of lemmas of potential independent interest is needed.

LEMMA 2. *If G is a spectrally dominant gmn on $M_n(C)$ and $A \in M_n(C)$ is such that $\rho(A)$ is attained by an eigenvalue λ of A whose Jordan block in the Jordan canonical form of A is not diagonal, then* $\rho(A) < G(A)$.

Proof. Without loss of generality we may assume that λ is a positive real number (because of the homogeneity of ρ and G) and that

$$A = \left(\begin{array}{c|c} \lambda I & 0 \\ \hline 0 & * \end{array}\right) + \left(\begin{array}{c|c} E & 0 \\ \hline 0 & 0 \end{array}\right),$$

where the two matrices on the right are partitioned conformally and E is the basic k-by-k nilpotent matrix defined in Example 2, $k \geqslant 2$. Let F denote the n-by-n matrix whose only nonzero entry is a 1 in the $k,1$ position (so that $F + E$ is the basic circulant permutation matrix), and let $t > 0$ be a real parameter. Then

$$\lambda + t^{1/k} = \rho(A + tF) \leqslant G(A + tF) \leqslant G(A) + tG(F).$$

Assume for the moment that $G(A)=\rho(A)=\lambda$; it would follow that

$$t^{(1-k)/k} \leqslant G(F) \qquad \text{for all} \quad t>0.$$

As $t \to 0$, $t^{(1-k)/k} \to \infty$, so that $G(F)$ would be unbounded, a contradiction. We conclude that actually $\rho(A)<G(A)$, as was to be shown. ■

LEMMA 3. *If G is a spectrally dominant gmn on $M_n(C)$, then for each $A \in M_n(C)$ there exists a (multiplicative) matrix norm M (which depends on A) such that*

$$M(A)=G(A).$$

Proof. If the hypothesis of Lemma 2 does not hold, we may choose S so that $S^{-1}AS$ is the Jordan canonical form of A modified so that any superdiagonal entries are small enough that the maximum absolute row sum is $\rho(A)$. Then for all $B \in M_n(C)$, define M by

$$M(B)=tM_1(S^{-1}BS),$$

where M_1 is the maximum absolute row sum norm $t=G(A)/\rho(A)$. Since $t \geqslant 1$, it is straightforward to verify that M is a (multiplicative) matrix norm, and $M(A)=G(A)$ by construction.

In case the hypothesis of Lemma 2 does hold, we may approximate $\rho(A)$ by the value of a (multiplicative) matrix norm (by a process similar to the above) arbitrarily closely. Since $\rho(A)<G(A)$ in this event, we may again choose M as desired, even though we may not choose M so that $M(A)=\rho(A)$. This completes the proof. ■

THEOREM 5. *Suppose that G is a gmn on $M_n(C)$. Then G is spectrally dominant if and only if for each $A \in M_n(C)$ there exists a constant γ_A (depending only on G and A) such that for all integers $k>0$,*

$$G(A^k) \leqslant \gamma_A G(A)^k.$$

Proof. If γ_A with the asserted properties exists for each $A \in M_n(C)$, then $\rho(A) \leqslant G(A)$ for each A by Theorem 2, and G is spectrally dominant.

Conversely, suppose G is spectrally dominant. Since G is a gmn, it is a vector norm on $M_n(C)$. If M is any other gmn, then, by the equivalence of vector norms [6], there exist positive constants α_M and β_M (depending on M)

such that

$$\alpha_M M \leqslant G \leqslant \beta_M M.$$

Consider $A \in M_n(C)$, choose an M guaranteed by Lemma 3, and let $\gamma_A = \beta_M$. We then have

$$G(A^k) \leqslant \gamma_A M(A^k) \leqslant \gamma_A M(A)^k = \gamma_A G(A)^k,$$

for all integers $k > 0$. This completes the proof of the existence of the desired γ_A. ∎

In a quite different way, the spectrally dominant generalized matrix norms are also characterized in [5].

EXAMPLE 3. Depending upon G (spectrally dominant) and on A, the γ_A guaranteed by Theorem 5 may have to be arbitrarily large. Let N be the maximum absolute value row sum matrix norm on $M_3(C)$, and let $t \geqslant 1$ be a parameter. Then for $A \in M_3(C)$ define G by

$$G(A) \equiv N(H \circ A)$$

where $\circ$ denotes the Hadamard (entrywise) product of matrices and

$$H = \begin{bmatrix} 1 & 1 & t \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Since $G \geqslant N$ and N is spectrally dominant, then G is spectrally dominant. However, if

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

then a short calculation shows that (the smallest) γ_A can be made arbitrarily large by choice of t. Two items should be noted, though. For this A and for any fixed t, $G(A^k) \leqslant G(A)^k$ for all k sufficiently large (depending on t). [See Theorem 6 below.] Also, for such a G and for any $A \in M_3(C)$, the maximum of $G(A^k)/G(A)^k$ occurs for $k \leqslant n$. This implies, for example, in this case that for a fixed G the constant γ_A in Theorem 5 may be chosen independent of A. (See Example 5 and following for further comments on this issue.)

If G is an induced matrix norm, it is well known [3] that

$$\lim_{k\to\infty} G(A^k)^{1/k} = \rho(A),$$

the existence of the limit being implicit.

REMARK. If G is any gmn on $M_n(C)$, then for all $A \in M_n(C)$,

$$\lim_{k\to\infty} G(A^k)^{1/k} = \rho(A). \tag{11}$$

Proof. Let M be an induced matrix norm, and choose α_M and β_M, positive real constants, as in the proof of Theorem 5, so that

$$\alpha_M M \leqslant G \leqslant \beta_M M. \tag{12}$$

Then for any $A \in M_n(C)$ and any positive integer k,

$$\left[\alpha_M M(A^k)\right]^{1/k} \leqslant G(A^k)^{1/k} \leqslant \left[\beta_M M(A^k)\right]^{1/k}. \tag{13}$$

Now, $\lim_{k\to\infty}[\alpha_M M(A^k)]^{1/k} = \lim_{k\to\infty}\alpha_M^{1/k}\lim_{k\to\infty}M(A^k)^{1/k} = 1\cdot\rho(A) = \rho(A)$, since the limit is known to exist and (11) is known to hold for an induced matrix norm. Similarly, $\lim_{k\to\infty}(\beta_M M(A^k))^{1/k} = \rho(A)$, and it follows from (13) that $\lim_{k\to\infty} G(A^k)^{1/k}$ exists and is equal to $\rho(A)$. ∎

EXAMPLE 4. Even if G is a spectrally dominant gmn, it may be that

$$G(A^k) > G(A)^k$$

for all positive integers $k \geqslant 2$. Thus no converse of Theorem 3 is possible. Let G_1 be the maximum absolute value row sum norm on $M_4(C)$, and suppose that G_2 is defined on $A = (a_{ij}) \in M_4(C)$ by $G_2(A) \equiv |a_{24}|$. Then

$$G \equiv G_1 + G_2$$

is a gmn on $M_4(C)$, and G is spectrally dominant, since $G \geqslant G_1$, which is spectrally dominant. However, if

$$A=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

then $G(A)=\rho(A)=1$, so that $G(A)^k=1$ for all positive integers k. But

$$A^k=\frac{1}{2^k}\begin{bmatrix} 2^k & 0 & 0 & 0 \\ 0 & 1 & k & \dfrac{k(k-1)}{2} \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

so that $G(A^k)=1+k(k-1)/2^{k+1}>1$ for $k\geqslant 2$. Thus $G(A^k)>G(A)^k$. ∎

The closest statement to a converse of Theorem 3 which we may make is then the following.

THEOREM 6. *If G is a spectrally dominant gmn on $M_n(C)$, then for each $A\in M_n(C)$, at least one of the following two statements must hold:*

(i) *There exists a positive integer $K(A)$ such that for all positive integers $k>K(A)$,*

$$G(A^k)\leqslant G(A)^k.$$

(ii) $\rho(A)=G(A)$.

Proof. Suppose that G is spectrally dominant, and it is enough to show that if

$$G(A^k)>G(A)^k$$

holds for infinitely many positive integers k, then $\rho(A)=G(A)$. But if $G(A^k)>G(A)^k$ infinitely often, then $G(A^k)^{1/k}>G(A)$ infinitely often, and it follows that

$$\lim_{k\to\infty} G(A^k)^{1/k}\geqslant G(A).$$

Since this limit equals $\rho(A)$ and since G is spectrally dominant, we have

$$\rho(A) \geqslant G(A) \geqslant \rho(A),$$

from which we may conclude that $\rho(A) = G(A)$. ∎

If attention is confined to rank 1 matrices, a stronger result than Theorems 3 and 6 may be obtained.

THEOREM 7. *If G is a gmn on $M_n(C)$, then G is spectrally dominant at all matrices of rank one if and only if*

$$G(A^2) \leqslant G(A)^2 \tag{14}$$

for all matrices A of rank one.

Proof. If (14) holds at all rank one matrices, then the spectral dominance of G at all rank ones follows via Lemma 1 in a manner similar to the proof of Theorem 3. On the other hand if G is spectrally dominant at all rank ones, then, since any rank one matrix $A \in M_n(C)$ may be written as $A = v_1 v_2^*$ where v_1 and v_2 are column vectors, we have

$$G(A) = G(v_1 v_2^*) \geqslant \rho(v_1 v_2^*) = |\mathrm{Tr}(v_1 v_2^*)| = |v_2^* v_1|.$$

It then follows that

$$G(A^2) = |v_2^* v_1| G(v_1 v_2^*) \leqslant G(A) G(v_1 v_2^*) = G(A)^2,$$

which means that (14) holds and completes the proof. ∎

We conclude with an example and some questions raised by the present work.

In each of the examples of a spectrally dominant gmn presented thus far, it has been the case that for all $A \in M_n(C)$ and any positive integer k,

$$\frac{G(A^k)}{G(A)^k} \leqslant \max_{1 \leqslant l \leqslant n} \frac{G(A^l)}{G(A)^l}.$$

In other words the minimum γ_A of Theorem 5 has been attained for $k \leqslant n$. That this is not, in general, the case for a spectrally dominant gmn is shown by the following.

EXAMPLE 5. On $M_2(C)$ define G by

$$G\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \max\{|a|+|b|,|c|+|d|\} + |c+d|.$$

Then G may be verified to be a gmn, and G is spectrally dominant because the maximum absolute row sum norm is. However, if

$$A = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix},$$

then $G(A)=1$, and, in general,

$$G(A^k) = \frac{2^k-1}{2^{k-1}}, \qquad k=1,2,\ldots.$$

Thus $G(A^k) \leqslant 2$ and $\gamma_A = 2$ is the smallest which satisfies (9), while the maximum in (15) is only $\frac{3}{2}$.

Further study of the class of spectrally dominant generalized matrix norms seems warranted. Among the questions which may be raised are the following:

(1) Which spectrally dominant gmn's do satisfy (15)? An answer to this question would provide interesting generalizations of some of the work in [1] and [2] and provide a positive special case answer to the next question.

(2) For each spectrally dominant gmn, can the γ_A of Theorem 5 be chosen independent of A? We conjecture that it can, and this is, by far, the most intriguing question. Understanding how the minimum γ depends upon G would then be important to the understanding of gmn's. Weak partial evidence for this conjecture was obtained in collaboration with Joel Anderson and is contained in the following extension of (part of) Theorem 5:

THEOREM 8. *Suppose that G is a generalized matrix norm on $M_n(C)$ satisfying $m(G)<1$. Then there exists a γ (depending only on G) such that for all $A \in M_n(C)$ and all positive integers k,*

$$G(A^k) \leqslant \gamma G(A)^k.$$

Proof. For each $A_0 \in M_n(C)$, apply Lemma 3 to the spectrally dominant gmn

$$H \equiv m(G)G,$$

thus producing a matrix norm M. Let β_M be a global constant (whose existence has already been noted) such that

$$G \leqslant \beta_M M.$$

Since $m(G) < 1$, it follows that

$$M(A) \leqslant G(A)$$

holds throughout some *neighborhood* $S(A_0)$ of A_0. In this neighborhood we have

$$G(A^k) \leqslant \beta_M M(A^k) \leqslant \beta_M M(A)^k \leqslant \beta_M G(A)^k.$$

Let $\gamma_{A_0} \equiv \beta_M$, and let U be the unit ball of G. For $A_0 \in U$, $\bigcup S(A_0)$ is an open cover of the necessarily compact set U. Therefore, by the Heine-Borel theorem, there is a finite subcover of U, each of whose elements is an $S(A_0)$. Letting γ be the maximum of γ_{A_0} over those A_0 which give this finite subcover proves the desired result by virtue of the homogeneity of G. ■

The question then is, simply, whether the conclusion of Theorem 8 still holds when $m(G) = 1$.

(3) For a gmn G, how does the minimum factor t which makes tG a matrix norm (i.e. multiplicative) relate to the spectral characteristic and also to the γ conjectured above?

(4) For a gmn G with $m(G) = 1$, what are the matrices A for which $\rho(A) = G(A)$?

(5) Assuming the conjecture in 2 above, how may the minimum γ be calculated for a given G? When is $\gamma = 1$? In view of [7] this may provide worthwhile numerical results.

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